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Laplacian and the Jacobi's inversion problem for the simple elliptic singularity

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§1. Introduction.

We construct *explicitly* (up to 1 unknown constant factor $\in \mathbf{C}^*$) the inversion mapping of the period mapping (for the primitive form) for the semi-universal deformation of the hypersurface simple elliptic singularity ($\tilde{E}_l$ type) by using the theta functions or the characters of an affine Lie algebra of type $E_l^{(1)}$ ($l = 6, 7, 8$).

§2. Review of the theory of period mapping for the primitive form.

In §2 and §3, we review the theory of primitive forms. For the notations and definitions, see [S]. Let Z, X, S, T δ_1, be a Hamiltonian system with the primitive form in the sense of Saito[S] obtained by the semi-universal deformation of the hypersurface simple elliptic singularity. We remind the notations:

$* : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$: commutative associative $\mathcal{O}_T$-algebra structure ,

$w : \mathcal{G} \longrightarrow Der_S(-logD)$,

$\nabla : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$: flat connection ,

$J : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{O}_T$: non-degenerate $\mathcal{O}_T$-bilinear form ,

$N : \mathcal{G} \longrightarrow \mathcal{G}$.

For our cases, n : the dimension of the Milnor fiber $= 2$, $r - 1$: the degree of the primitive form $= 0$, μ : the Milnor number of the $\tilde{E}_l$ type ($l = 6, 7, 8$) simple elliptic singularity

$= l+2$ in the notation of [S]. The exponents for the primitive form are

$$\begin{aligned}
&\tilde{E}_6 \text{ case}: 1, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, 2,\\
&\tilde{E}_7 \text{ case}: 1, \frac{5}{4}, \frac{5}{4}, \frac{6}{4}, \frac{6}{4}, \frac{6}{4}, \frac{7}{4}, \frac{7}{4}, 2,\\
&\tilde{E}_8 \text{ case}: 1, \frac{7}{6}, \frac{8}{6}, \frac{8}{6}, \frac{9}{6}, \frac{9}{6}, \frac{10}{6}, \frac{10}{6}, \frac{11}{6}, 2.
\end{aligned}$$

The period mapping in the usual sense (integral of the primitive form) degenerates by the existence of the integral exponents. In fact, it maps the $\mathbf{C}^*$-orbit in S to 1 point. In order to construct the reasonable period mapping (i.e. separating the different points in the $\mathbf{C}^*$-orbit), we need to add a new function as a period. This was done (Saito[S]) in the following formulation:

$$\begin{aligned}
&M^{(s)} := \mathcal{D}_S / \sum_{\delta,\delta'\in\mathcal{G}} \mathcal{D}_S P(\delta,\delta') + \sum_{\delta\in\mathcal{G}} \mathcal{D}_S Q_s(\delta) \quad (s\in\mathbf{C}),\\
&P(\delta,\delta') := \delta\delta' - (\delta * \delta')\delta_1 - \nabla_\delta \delta' \ (\delta,\delta'\in\mathcal{G}),\\
&Q_s(\delta) := w(\delta)\delta_1 - (N-s-1)\delta \ (\delta\in\mathcal{G}),\\
&Sol(M^{(s)}) := Hom_{\mathcal{D}_S}(M^{(s)},\mathcal{O}_S) \quad (s\in\mathbf{C}).
\end{aligned}$$

Then the morphism: $\mathcal{D}_S \longrightarrow M^{(s)} \longrightarrow 0$ induces

$$0 \longrightarrow Sol(M^{(s)}) \longrightarrow \mathcal{O}_S.$$

The exterior derivative : $d:\mathcal{O}_S \longrightarrow \Omega^1_S$ induces

$$dSol(M^{(s)}) \longrightarrow \Omega^1_S.$$

For $p_0 \in S\setminus D$, the period mapping :

$$P:\tilde{S} \longrightarrow E := \{x \in \mathrm{Hom}_{\mathbf{C}}(Sol(M^{(1)})_{p_0},\mathbf{C}) | x(1_S)=1,\ \mathrm{Im}\ x(\tau)>0\},$$

(for the definitions of P and $\tilde{S}$, see [S]) gives the isomorphism of the analytic spaces, and is equivariant with the monodromy group action. Here the following diagram holds:

$$\begin{array}{ccc}
\widetilde{S\setminus D} & \subset & \tilde{S}\\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi}\\
S\setminus D & \subset & S
\end{array},$$

where $\widetilde{S \setminus D}$ is a monodromy covering of $S \setminus D$ for the local system $Sol(M^{(1)})|_{S\setminus D}$. We remark that $\tilde{S} \setminus (\widetilde{S \setminus D})$ is a divisor in $\tilde{S}$. $\tau \in Sol(M^{(1)})$ is a degree 0 flat coordinate given by the classical period.

§3. Prepotential and the definition of the tensor I.

In order to study the period mapping P, we review the integrable structures on S.

Proposition 3.1. *(Saito [S] see also Matsuo[M]).*

1) *The following $\mathcal{F} \in \mathcal{O}_S$ exists: $v_1 v_2 v_3 \mathcal{F} = J(v_1 * v_2, v_3)$ for $v_i \in \mathcal{G}$ s.t. $\nabla v_i = 0$ (horizontal section). We call $\mathcal{F}$ the prepotential.*
2) *The prepotential $\mathcal{F}$ satisfies the WDVV equations.*

Proposition 3.2. *(Saito [S]).*

1) *The following :*

$$I : \Omega^1_S \times \Omega^1_S \longrightarrow \mathcal{O}_S : (\omega, \omega') \mapsto \sum_{i=0}^{\mu-1} < \delta_i \omega >< w(\delta^{i*}), \omega' >,$$

gives a non-degenerate symmetric $\mathcal{O}_S$-bilinear form, where δ_i, δ^{i*} are both $\mathcal{O}_T$-free basis of $\mathcal{G}$ s.t. $J(\delta_i, \delta^{j*}) = \delta_{ij}$, w is a morphism : $w : \mathcal{G} \longrightarrow Der_S(-logD)$ introduced in §2.

2) *By the morphism $dSol(M^{(1)})|_{S\setminus D} \longrightarrow \Omega^1_S|_{S\setminus D}$, the following is induced :*

$$I : dSol(M^{(1)})|_{S\setminus D} \times dSol(M^{(1)})|_{S\setminus D} \longrightarrow \mathbf{C}_{S\setminus D}.$$

This gives a non-degenerate symmetric $\mathbf{C}_{S\setminus D}$-bilinear form.

3) *I induces the $\mathbf{C}$-bilinear form I_{p_0} on $(dSol(M^{(1)}))_{p_0}$. Since the cotangent space of E is canonically identified with $(dSol(M^{(1)}))_{p_0}$, I_{p_0} defines an $\mathcal{O}_E$-bilinear form:*

$$I_E : \Omega^1_E \times \Omega^1_E \longrightarrow \mathcal{O}_E.$$

4) *By the period mapping: $P : \tilde{S} \longrightarrow E$, we have*

$$P^* I_E = \psi^* I.$$

Proposition 3.3. *(Saito [S][S1]). The space E with the monodromy group acion and with the tensor $(2\pi\sqrt{-1})^{-2}I_E$ for the singularity of type $\tilde{E}_l$ is identified with $\tilde{\mathbf{E}}$, hyperbolic Weyl group $\tilde{W}_R$, and the tensor $\tilde{I}_{\tilde{\mathbf{E}}}$ of the elliptic root system of type $E_l^{(1,1)}$.*

§4. Laplacian and the Jacobi's inversion problem.

In this section we take the flat coordinates $t_0, \cdots, t_{\mu-1}$ s.t. $\frac{\partial}{\partial t_0} = \delta_1$. We call $f \in \mathcal{O}_S$ homogeneou of degree ν if $Ef = \nu f$ for the Euler vector field $E := w(\delta_1)$ and denote $\nu = degf$. We also assume that t_i are homogeneous.

Notation. *$\eta_{ij} := J(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}) \in \mathbf{C}$, η^{ij} is defined by the equations : $\eta_{ij}\eta^{jk} = \delta_i^k$ (Kronecker's delta).*

Proposition 4.1. *Let $\mathcal{F}$ be a prepotential. We assume that $\mathcal{F}$ is homogeneous. Then*

1) The tensor I on S is written as follows:

$$I(dt_i, dt_j) = \frac{degt_i + degt_j}{degt_0} \sum_{p,q=0}^{\mu-1} \eta^{ip}\eta^{jq} \frac{\partial}{\partial t_p}\frac{\partial}{\partial t_q}\mathcal{F},$$

2) The Laplacian D on $\mathcal{O}_S$ for the tensor I on S is written as follows:

$$D = \sum_{i,j=0}^{\mu-1} I(dt_i, dt_j)\frac{\partial^2}{\partial t_i \partial t_j} + \sum_{k,m=0}^{\mu-1} \frac{degt_k}{degt_0}\eta^{km}\frac{\partial Tr}{\partial t_m}\frac{\partial}{\partial t_k},$$

where $Tr := \sum_{i,j=0}^{\mu-1} \eta_{ij} I(dt_i, dt_j) = \sum_{i,j=0}^{\mu-1} \eta^{ij}\frac{\partial}{\partial t_i}\frac{\partial}{\partial t_j}\mathcal{F}$.

3) The twisted Laplacian D_{A_ρ} on $\mathcal{O}_S$ is written as follows: for homogeneous $f \in \mathcal{O}_S$,

$$D_{A_\rho}(f) = D(f) + \sum_{i,j=0}^{\mu-1} \eta^{ij}\frac{\partial Tr}{\partial t_j}\frac{\partial f}{\partial t_i} - \frac{degf}{degA_\rho}\frac{1}{2}\sum_{i,j=0}^{\mu-1}\eta^{ij}\frac{\partial^2 Tr}{\partial t_i \partial t_j} f,$$

where

$$D_{A_\rho}(f) := A_\rho^{-1} D(A_\rho f),$$

$$A_\rho := (\text{ unit of } \mathcal{O}_S) \times \Delta^{1/2} \ s.t. D(A_\rho) = 0,$$

$$\Delta := det((I(dt_i, dt_j))_{i,j=0,\cdots,\mu-1}).$$

A_ρ is normalized up to constant factor by the condition $D(A_\rho) = 0$, thus D_{A_ρ} is well defined.

Since the prepotential can be calculated by the results of Noumi([N]) or the results of Verlinde-Warner ([V-W]) etc., we can calculate the Laplacian and the following: we follow the notations of t_i as in [K-T-S].

Proposition 4.2.

1) ($\tilde{E}_6$ case) Let t_4, t_5, t_6 be the lowest non-zero degree flat coordinates (degree 1/3). Then the other flat coordinates t_0, t_1, t_2, t_3 can be explicitly written as a polynomial of t_4, t_5, t_6 with coefficients of the known degree 0 function and Laplacian D (resp. D_{A_ρ}).

2) ($\tilde{E}_7$ case) Let t_6, t_7 be the lowest non-zero degree flat coordinates (degree 1/4). Then the other flat coordinates $t_0, t_1, t_2, t_3, t_4, t_5$ can be explicitly written as a polynomial of t_6, t_7 with coefficients of the known degree 0 function and Laplacian D (resp. D_{A_ρ}).

3) ($\tilde{E}_8$ case) Let t_8 be the lowest non-zero degree flat coordinates (degree 1/6). Then the other flat coordinates $t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7$ can be explicitly written as a polynomial of t_8 with coefficients of the known degree 0 function and Laplacian D (resp. D_{A_ρ}).

Also the action of the Laplacian on the lowest non-zero degree flat coordinates relates them with the theta functions or characters of an affine Lie algebras on the space $E \simeq \mathbf{H} \times \mathfrak{h}_{\mathbf{C}} \times \mathbf{C}$ (where $\mathbf{H} := \{z \in \mathbf{C} | \mathrm{Im}\ z > 0\}$, $\mathfrak{h}_{\mathbf{C}}$: complex Cartan subalgebra for E_l type for $\tilde{E}_l$ type singularity). For the definition of Θ_Λ: theta function and $\frac{A_{\Lambda+\rho}}{A_\rho}$: normalized character for the integrable irreducible highest weight module (they are holomorphic functions on $\mathbf{H} \times \mathfrak{h}_{\mathbf{C}} \times \mathbf{C}$), see Kac[K].

Proposition 4.3. *By choosing the suitable primitive form and the suitable identification of E with $\mathbf{H} \times \mathfrak{h}_{\mathbf{C}} \times \mathbf{C}$ (which could be calculable), we have*

1) ($\tilde{E}_6$ case)

$$t_4 = c\eta^{-8}\Theta_{\Lambda_0} = c\eta^{-2}\frac{A_{\Lambda_0+\rho}}{A_\rho},$$

$$t_5 = c\eta^{-8}\Theta_{\Lambda_1} = c\eta^{-2}\frac{A_{\Lambda_1+\rho}}{A_\rho},$$
$$t_6 = c\eta^{-8}\Theta_{\Lambda_5} = c\eta^{-2}\frac{A_{\Lambda_5+\rho}}{A_\rho},$$

where $c \in \mathbf{C}^*$ is a non-zero constant.

2) ($\tilde{E}_7$ case) By choosing the suitable primitive form, we have

$$t_6 = c'\eta^{-9}\Theta_{\Lambda_0} = c'\eta^{-2}\frac{A_{\Lambda_0+\rho}}{A_\rho},$$
$$t_7 = c'\eta^{-9}\Theta_{\Lambda_6} = c'\eta^{-2}\frac{A_{\Lambda_6+\rho}}{A_\rho},$$

where $c' \in \mathbf{C}^*$ is a non-zero constant.

3) ($\tilde{E}_8$ case) By choosing the suitable primitive form, we have

$$t_8 = c''\eta^{-10}\Theta_{\Lambda_0} = c''\eta^{-2}\frac{A_{\Lambda_0+\rho}}{A_\rho},$$

where $c'' \in \mathbf{C}^*$ is a non-zero constant.

For the proof, we first use the characterization of the theta function and the character of an affine Lie algebra respectively. Under the identificattion of the proposition 4.3. and by, $(\psi \circ P^{-1})^*$, we have :

$$\left(\bigoplus_{\Lambda:level\ k} \mathbf{C}\Theta_\Lambda\right)^{W_{E_l}} = \mathrm{ker}D \cap \{f \in \Gamma(S, \mathcal{O}_S) | degf = k/m_l\},$$
$$\bigoplus_{\Lambda:level\ k} \mathbf{C}\frac{A_{\Lambda+\rho}}{A_\rho} = \mathrm{ker}D_{A_\rho} \cap \{f \in \Gamma(S, \mathcal{O}_S) | degf = k/m_l\},$$

where m_l is an integer corresponding to $\tilde{E}_l$ defined by $m_6 = 3, m_7 = 4, m_8 = 6$, W_{E_l} is a Weyl group of type E_l. For $k = 1$ in the above, the calculation of the action of the Laplacian on the lowest degree non-zero flat coordinates gives

$$(RHS) = (\text{ known degree 0 function }) \times V, \qquad (*)$$

where $V :=$ the linear span of the lowest degree non-zero flat coordinates. Moreover by using the equivalence of the period mapping under the automorphism group action of the Hamiltonian system, we have the equality $(*)$ as an irreducible module for the automorphism group action of the Hamiltonian system. This gives the correspondence of the flat coordinates and the theta function (resp. the character) up to constant factor.

These propositions enable us to express the flat coordinates by the theta functions or character of an affine Lie algebra. Since

1) all flat coordinates are expressed by the Laplacian (resp. the twisted Laplacian) and the lowest degree non-zero flat coordinates,
2) the lowest degree non-zero flat coordinates are expressed by the theta functions (resp. character of an affine Lie algebra),
3) $(2\pi\sqrt{-1})^{-2}D$ can be identified with the Laplacian for elliptic root system (or the ones for affine Lie algebras) so its action on theta functions can be calculated. Also A_ρ can be identified (up to constant) with the Weyl-Kac denominator, so the action of $(2\pi\sqrt{-1})^{-2}D_{A_\rho}$ on the character or the products of the character can be calculated by using the tensor product expansion of the representations of the irreducible highest weight modules.

§5. Example.

$\tilde{E}_6$ case : We choose the flat coordinates $t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7 = t = \tau$ (where τ is a function introduced in §2 and is just the uniformizing parameter of the modulus of the elliptic curve which appear in the compactification of the Milnor fiber) s.t. the semi-universal deformation of $\tilde{E}_6$ singularity is given by the following equation:

$$\begin{aligned} W = &-\frac{1}{3}(x_1^3 + x_2^3 + x_3^3) + \alpha_1(t)(x_1x_2x_3) + \alpha_2(t)(t_4x_1x_2 + t_5x_1x_3 + t_6x_2x_3) \\ &+ \alpha_3(t)(t_1x_1 + t_2x_2 + t_3x_3) + \alpha_4(t)(t_4t_5x_1 + t_4t_6x_2 + t_5t_6x_3) \\ &+ \frac{1}{2}\alpha_5(t)(t_6^2x_1 + t_5^2x_2 + t_4^2x_3) + \alpha_6(t)(t_1t_6 + t_2t_5 + t_3t_4) \\ &+ \frac{1}{6}\alpha_7(t)(t_4^3 + t_5^3 + t_6^3) + \alpha_8(t)t_4t_5t_6 + t_0, \end{aligned}$$

where

$$\begin{aligned} \alpha_1(t) &= \alpha, \\ \alpha_2(t) &= (\alpha')^{1/2}(1-\alpha^3)^{1/6}, \\ \alpha_3(t) &= (\alpha')^{1/2}(1-\alpha^3)^{-1/6}, \\ \alpha_4(t) &= -\alpha^2\alpha'(1-\alpha^3)^{-2/3}, \\ \alpha_5(t) &= -\alpha\alpha'(1-\alpha^3)^{-2/3}, \\ \alpha_6(t) &= -\frac{1}{2}(\frac{\alpha''}{\alpha'} + \frac{3\alpha^2\alpha'}{1-\alpha^3}), \\ \alpha_7(t) &= -(\alpha')^{3/2}(1-\alpha^3)^{-1/2}, \\ \alpha_8(t) &= -\alpha(\alpha')^{3/2}(1-\alpha^3)^{-1/2}, \end{aligned}$$

and $'$ means $\frac{1}{3(-2\pi\sqrt{-1})}\frac{d}{d\tau}$ Then these flat coordinates give the the following monodromy group invariant holomorphic functions on the period domain E:

$$\begin{aligned} t_4 &= c\eta^{-8}\Theta_{\Lambda_0} = c\eta^{-2}\frac{A_{\Lambda_0+\rho}}{A_\rho}, \\ t_5 &= c\eta^{-8}\Theta_{\Lambda_1} = c\eta^{-2}\frac{A_{\Lambda_1+\rho}}{A_\rho}, \\ t_6 &= c\eta^{-8}\Theta_{\Lambda_5} = c\eta^{-2}\frac{A_{\Lambda_5+\rho}}{A_\rho}, \end{aligned}$$

$$t_1 = c^2\frac{1}{\alpha}\left(\frac{\alpha'}{1-\alpha^3}\right)^{-1/2}$$
$$\times\left[\frac{3}{4}D(\eta^{-16}\Theta^2_{\Lambda_5}) + (\frac{5}{2}\frac{\alpha''}{\alpha'} + 4\frac{\alpha^2\alpha'}{1-\alpha^3})\eta^{-16}\Theta^2_{\Lambda_5} - \frac{\alpha'}{1-\alpha^3}\eta^{-16}\Theta_{\Lambda_0}\Theta_{\Lambda_1}\right]$$
$$= c^2\frac{1}{\alpha}\left(\frac{\alpha'}{1-\alpha^3}\right)^{-1/2}$$
$$\times\left[\frac{3}{4}D_{A_\rho}(\eta^{-4}\left(\frac{A_{\Lambda_5+\rho}}{A_\rho}\right)^2) + (7\frac{\alpha''}{\alpha'} + \frac{43}{4}\frac{\alpha^2\alpha'}{1-\alpha^3})\eta^{-4}\left(\frac{A_{\Lambda_5+\rho}}{A_\rho}\right)^2 - \frac{\alpha'}{1-\alpha^3}\eta^{-4}\frac{A_{\Lambda_0+\rho}}{A_\rho}\frac{A_{\Lambda_1+\rho}}{A_\rho}\right],$$
$$t_2, t_3 = \text{ change of the suffix of } t_1,$$
$$t_0 = \frac{1}{6}\left[D(t_1t_6 + t_2t_5 + t_3t_4) - \sum_{i=1}^{6} t_i\frac{\partial Tr}{\partial t_i}\right],$$
$$Tr = 8t_0 + (t_1t_6 + t_2t_5 + t_3t_4)(-2\frac{\alpha''}{\alpha'} - 3\frac{\alpha^2\alpha'}{1-\alpha^3})$$
$$+\frac{1}{6}(t_4^3 + t_5^3 + t_6^3)(2+\alpha^3)(\frac{\alpha'}{1-\alpha^3})^{3/2} + 3t_4t_5t_6\alpha(\frac{\alpha'}{1-\alpha^3})^{3/2},$$

We remark that RHS of the equation of t_0 contains only $t_1, \cdots, t_6$, so substituting the equations above, we obtain the expression in terms of theta functions. Since the difference of D and D_{A_ρ} is written by Tr and is obtained by the above, we obtain the expression of t_0 in terms of $t_1, \cdots, t_6$ and D_{A_ρ}, thus obtain the expression of t_0 in terms of character and D_{A_ρ}.

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